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High-precision evaluation of four-loop vacuum bubbles in three dimensions

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Abstract

In this letter we present a high-precision evaluation of the expansions in $\epsilon = (3-d)/2$ of (up to) four-loop scalar vacuum master integrals, using the method of difference equations developed by Laporta. We cover the complete set of fully massive master integrals.

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1 Introduction

Higher-order perturbative computations have become a necessity in many areas of theoretical physics, be it for high-precision tests of QED, QCD and the standard model, or for studying critical phenomena in condensed matter systems.

Most recent investigations employ a highly automated approach, utilizing algorithms that can be implemented on computer algebra systems, in order to handle the growing numbers of diagrams as well as integrals which occur at higher loop orders.

Computations can be divided into four key steps. First, the complete set of diagrams including symmetry factors has to be generated. For a detailed description of an algorithm for this step for the case of vacuum topologies, see [1]. Second, after specifying the Feynman rules, the color- and Lorentz-algebra has to be worked out. Third, within dimensional regularization, massive use of the integration-by-parts (IBP) technique [2] to derive linear relations between different Feynman integrals in conjunction with an ordering prescription can be used to reduce the (typically large number of) integrals to a basis of (typically a few) master integrals [3]. Practical notes as well as a classification of vacuum master integrals is given in [4]. Fourth, the master integrals have to be solved, either fully analytically, or in an expansion around the space-time dimension d of interest.

It is the fourth step that we wish to address here. While most work has been and is being devoted to $d = 4$, perturbative results in lower dimensions are needed for applications in condensed matter systems, as well as in the framework of dimensionally reduced effective field theories for thermal QCD, where recent efforts have made four-loop contributions an issue [5].

A very important subset of master integrals are fully massive vacuum (bubble) integrals, since they constitute a main building block in asymptotic expansions (see e.g. [6]). They are also useful for massless

theories, when a propagator mass is introduced as an intermediate infrared regulator [7].

The main purpose of this note is to numerically compute the complete set of fully massive vacuum master integrals in terms of a high-precision ϵ -expansion in $d = 3 - 2\epsilon$ dimensions, in complete analogy with the four-dimensional work of S. Laporta [8].

The plan of the paper is as follows. In Section 2, we give a brief review of the method of difference equations applied to vacuum integrals. In Section 3, we discuss the actual implementation of the algorithm. In Section 4, we display our numerical results for the truncated power series expansions in ϵ of all fully massive master integrals, up to four-loop level, in $d = 3 - 2\epsilon$.

2 The evaluation of master integrals through difference equations

The method we have chosen to compute the coefficients of the truncated power series expansions of the master integrals is based on constructing difference equations for the integrals and then solving them numerically using factorial series. This approach was recently developed in Ref. [3], and below we briefly summarize its basic concepts following the notation of the original paper, which contains a much more detailed presentation on the subject. While the method is completely general as it applies to arbitrary kinematics, masses and topologies [9], our brief summary is somewhat adapted to the specific case of massive vacuum integrals.

The main idea is to attach an arbitrary power x to one of the lines of a master integral U ,

$$U(x) \equiv \int \frac{1}{D_1^x D_2 \dots D_N}, \quad (1)$$

where the $D_i = (p_i^2 + 1)$ denote inverse scalar propagators. In our case all of these share the same mass m , which we have therefore set to 1, noting that it can be restored in the end as a trivial dimensional prefactor of each integral. The original integral is then just $U = U(1)$. Depending on the symmetry properties of the integral, there can be different choices for the ‘special’ line with the arbitrary power x , but in the limit $x = 1$ they all reduce to the original integral U . This degeneracy can (and will later) be used for non-trivial checks of the method.

Employing IBP identities in a systematic way, it is possible to derive a linear difference equation obeyed by the generalized master integral $U(x)$,

$$\sum_{j=0}^R p_j(x) U(x+j) = F(x), \quad (2)$$

where R is a finite positive integer and the coefficients p_j are polynomials in x (and the space-time dimension d). The function F on the r.h.s. is a linear combination of functions analogous to $U(x)$ but derived from simpler master integrals, i.e. integrals containing a smaller number of loops and/or propagators.

The general solution of this kind of an equation is the sum of a special solution of the full equation, $U_0(x)$, and all solutions of the homogeneous equation ($F = 0$),

$$U(x) = U_0(x) + \sum_{j=1}^R U_j(x), \quad (3)$$

where each ($j = 0, \dots, R$)

$$U_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)} \quad (4)$$

is a factorial series¹. Substituting into Eq. (2), one obtains the coefficients μ and K (the latter being a function of d), as well as recursion relations for the x -independent coefficients $a(s)$ (being functions of d as

¹For a rigorous definition of the concept as well as a motivation for this kind of an ansatz, we refer the reader to Ref. [3].

well) for each solution. For the homogeneous solutions, these recursion relations relate all coefficients to their value at $s = 0$, $a_j(s) = c_j(s) a_j(0)$, where the $c_j(s)$ are rational functions (of d as well). For the special solution, the $a_0(s)$ are completely fixed in terms of the inhomogeneous part $F(x)$, consisting of ‘simpler’ integrals which are assumed to already be known in terms of their factorial series expansions.

What remains to be done is to fix the x - and s -independent constants $a_j(0)$, $j \neq 0$, in order to determine the weights of the different homogeneous solutions. To this end, it is most useful to study the behavior of $U(x)$ at large x , where the first factor in

$$U(x) = \int \frac{1}{(p_1^2 + 1)^x} g(p_1) \quad (5)$$

peaks strongly around $p_1^2 = 0$. Hence, the large- x behavior of the modified master integral is determined by the small-momentum expansion of the two-point function $g(p_1)$, which has one loop less than the original vacuum integral. In fact, for all cases we cover here, the first coefficient in the asymptotic expansion suffices. This is furthermore particularly simple, since it factorizes into a one-loop bubble carrying the large power x and a lower-loop vacuum bubble $g(0)$, which corresponds to $U(x)$ with its ‘special’ line cut away,

$$\lim_{x \rightarrow \infty} U(x) = \left[\int \frac{1}{(p_1^2 + 1)^x} \right] \times [g(0)] \sim (1)^x x^{-d/2} g(0). \quad (6)$$

A comparison with the large- x behavior of Eqs. (3), (4), proportional to $\sum_j \mu_j^x a_j(0) x^{K_j}$, can now be used to fix the $a_j(0)$, of which maximally one will turn out to be non-zero for our set of integrals.

Having the full solution at hand, we have in principle completed our entire task, as in the limit $x = 1$ we recover from $U(x)$ the value of the initial integral. Let us, however, add a couple of practical remarks here. What is still to be done is to perform the summation of the factorial series of Eq. (4), which means truncating the infinite sum at some s_{\max} . Studying the convergence behavior of these sums, one notices that even in the cases where they do converge down to $x \sim 1$, their convergence properties usually strongly decline with decreasing x . This means that in practical computations, where one aims at obtaining a maximal number of correct digits for $U(1)$ with as little CPU time as possible, the optimal strategy is to evaluate the integral $U(x)$ with the factorial series approach at some $x_{\max} \gg 1$ and then use the recurrence relation of Eq. (2) to obtain the desired result at $x = 1$. The price to pay is, however, a loss of numerical accuracy at each ‘pushdown’ ($x \rightarrow x - 1$) step due to possible cancellations, which makes the use of a very high x_{\max} impossible. In practice the strategy is to determine an optimal value for the ratio s_{\max}/x_{\max} . To give an example, for the four-loop integrals of Section 4 we have found that $s_{\max}/x_{\max} \sim 50$ is a good value, while we used a range of $s_{\max} \sim 1350 \dots 2000$.

3 Implementation of the algorithm

As is apparent from the preceding section, there are three main steps involved in obtaining the desired numerical coefficients in the ϵ -expansion of each master integral: deriving the difference equations obeyed by each integral, solving them in terms of factorial series, and finally performing the ϵ -expansion and numerically evaluating the sum of Eq. (4) (truncated at s_{\max}) to the precision needed. We will briefly address each of them in the following.

For the first step, we slightly generalized the IBP algorithm we had used for reducing generic 4-loop bubble integrals to master integrals, which follows the setup given in [3], and whose implementation in FORM [10] is documented in [4]. The main difference is an enlarged representation for the integrals, keeping track of the line which carries the extra powers x , as well as the fact that there are now two independent variables (d, x), requiring factorization (and inversion) of bivariate polynomials, as opposed to univariate polynomials in the original version.

Second, staying within FORM for convenience, we implemented routines that straightforwardly solve the difference equations in terms of factorial series, along the lines of [3]. This is done starting with the

$$-993306.5068744076465770453 \epsilon^5 - 1406349.173668893367086333 \epsilon^6 + \mathcal{O}(\epsilon^7) \quad (14)$$

$$\begin{aligned}
= & + 8.000000000000000000000000 \epsilon^{-2} - 25.94976691892252044659284 \epsilon^{-1} \\
& - 152.5193565764658289654545 + 2653.873458838396323815566 \epsilon \\
& - 23471.05910309626447406639 \epsilon^2 + 169839.2007120049515774452 \epsilon^3 \\
& - 1124117.877397355450165203 \epsilon^4 + 7116455.837989754857686241 \epsilon^5 + \mathcal{O}(\epsilon^6) \quad (15)
\end{aligned}$$

$$\begin{aligned}
= & + 78.95683520871486895067593 \, \epsilon^{-1} - 1062.608419332108844057560 \\
& + 9340.076804859596283223881 \, \epsilon - 68699.47293187699594375521 \, \epsilon^2 \\
& + 462145.6926820632806821051 \, \epsilon^3 - 2963063.672524354359852913 \, \epsilon^4 \\
& + 18494675.22629230338091457 \, \epsilon^5 - 113673206.9834859509114931 \, \epsilon^6 + \mathcal{O}(\epsilon^7) \quad (16)
\end{aligned}$$

$$\begin{aligned}
= & + 33.05150971425671642138224 - 358.4595946559340238066389 \epsilon \\
& + 2451.469078369636793421997 \epsilon^2 - 13564.14170819716549262162 \epsilon^3 \\
& + 66602.55178881628657891800 \epsilon^4 - 303915.1384697444382333780 \epsilon^5 \\
& + 1323370.670112542076081095 \epsilon^6 - 5589978.086026239748023404 \epsilon^7 + \mathcal{O}(\epsilon^8) \quad (17)
\end{aligned}$$

$$\begin{aligned}
= & + 27.57584879577521927818358 - 291.4075344540614879796315 \epsilon \\
& + 1956.162997112043390446958 \epsilon^2 - 10678.5639091187201818981 \epsilon^3 \\
& + 51925.3888799007705970928 \epsilon^4 - 235296.36309585614167636 \epsilon^5 \\
& + 1019555.9650538012793966 \epsilon^6 - 4292011.3101269758990557 \epsilon^7 + \mathcal{O}(\epsilon^8)
\end{aligned} \tag{18}$$

$$\begin{aligned}
= & + 19.84953756526739935782082 - 200.9768306606422068619864 \epsilon \\
& + 1308.883448000100198800887 \epsilon^2 - 6990.22562100063537185149 \epsilon^3 \\
& + 33456.8326902483214417013 \epsilon^4 - 149903.697032731221510018 \epsilon^5 \\
& + 644404.61801211590204150 \epsilon^6 - 2697912.0878890801856234 \epsilon^7 + \mathcal{O}(\epsilon^8)
\end{aligned} \tag{19}$$

$$\begin{aligned}
= & + 3.141336279450209755917806 - 19.78740273338730374386071 \, \epsilon \\
& + 83.81604328128850410126511 \, \epsilon^2 - 295.3496021971085625102731 \, \epsilon^3 \\
& + 934.2247995435558122394582 \, \epsilon^4 - 2751.31852347627462886909 \, \epsilon^5 \\
& + 7700.18972963585089750348 \, \epsilon^6 - 20740.9769474365145116212 \, \epsilon^7 + \mathcal{O}(\epsilon^8) \quad (20)
\end{aligned}$$

$$\begin{aligned}
= & + 2.012584635078182771827701 - 10.76814227797251921324485 \epsilon \\
& + 39.40636857271936487899035 \epsilon^2 - 121.0015646826735646109733 \epsilon^3 \\
& + 335.6942965583773421544251 \epsilon^4 - 872.009773755552224781319 \epsilon^5 \\
& + 2163.88707221986880315576 \epsilon^6 - 5193.51249188593850483093 \epsilon^7 + \mathcal{O}(\epsilon^8) \quad (21)
\end{aligned}$$

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$$+ 549.30166596161426941705 \epsilon^6 - 1164.5120588971521623546 \epsilon^7 + \mathcal{O}(\epsilon^8) \quad (22)$$

$$\begin{aligned} \text{Diagram 1} &= + 0.297790726683752651865168 - 0.709896385699143430126726 \epsilon \\ &+ 1.40535549472683132370135 \epsilon^2 - 2.45721908509256673440117 \epsilon^3 \\ &+ 4.00998036005764459707090 \epsilon^4 - 6.2518071963546459390185 \epsilon^5 \\ &+ 9.4402506572040685160665 \epsilon^6 - 13.924465979877416801887 \epsilon^7 + \mathcal{O}(\epsilon^8) \end{aligned} \quad (23)$$

$$\begin{aligned} \text{Diagram 2} &= + 0.233923932580303206470057 - 0.48523164074102176840584 \epsilon \\ &+ 0.88555744401503729577888 \epsilon^2 - 1.438019871368410241810 \epsilon^3 \\ &+ 2.198725350440790755608 \epsilon^4 - 3.231974794381719679729 \epsilon^5 + \mathcal{O}(\epsilon^6) \end{aligned} \quad (24)$$

$$\begin{aligned} \text{Diagram 3} &= + 0.195906401341238799905792 - 0.37006152907989745845214 \epsilon \\ &+ 0.65228273818146302130509 \epsilon^2 - 1.029288152514143871118 \epsilon^3 \\ &+ 1.542484509438506710808 \epsilon^4 + \mathcal{O}(\epsilon^5) \end{aligned} \quad (25)$$

We have performed various checks in order to test the correctness of our recursion relations as well as to verify the number of exact digits contained in our results Eqs. (8)-(25). The first task we have completed by exploiting the fact that the recursion relations are not specific to $d = 3 - 2\epsilon$, but can easily be applied to any dimension, such as $d = 4 - 2\epsilon$. We have successfully verified the results of Ref. [8] to somewhat lower accuracy and depth in ϵ . Note that our choice of a basis for 4-loop master integrals differs slightly from the one made in [8]. The relations needed for a basis transformation are listed in [4]. An immediate advantage in the light of difference equations is that with our choice, the above results Eqs. (14),(20) and (23) follow ‘for free’ from their counterparts without dots.

The accuracy of our three-dimensional results we have on the other hand examined in three independent ways:

- by comparing the numerical results to existing analytic calculations; they can be found in [11] (divergent and constant parts of Eqs. (9)-(11)), [12] (leading term of Eq. (12)), [13] (divergence of Eq. (16)) and [14,15] (all divergences and some constant parts of 4-loop integrals, as well as some $\mathcal{O}(\epsilon)$ terms of lower-loop cases).
- by comparing the results obtained by raising topologically inequivalent lines to the power x ,
- by analyzing the convergence properties of the factorial series, i.e. by checking the stability of our results with respect to varying s_{\max} .

The first method is of course exact, but is only available for a few low (in ϵ) orders for approximately half of the integrals considered. The second one, on the other hand, has the advantage of covering all the different powers of ϵ , but is inapplicable for those integrals, in which all propagators are equivalent (e.g. the basketball-topology). The third method is then the most widely applicable one, but has the downside of providing no evidence for the correctness of our results, rather giving only the number of digits stable in the variation of the cut-off of the factorial series. For the integral of Eq. (25) only the last method is available, but in addition we have verified the leading term in the result to 3 digits using a Monte Carlo integration of an 8-dimensional integral representation derived for this integral in Ref. [14].

One might be concerned about the rapid growth with increasing ϵ -orders of most of the coefficients. This is, as was pointed out in [8], caused by poles that the integrals (seen as functions of d) develop near $d = 3$, e.g. at $d = 7/2, 4$, etc. It is to be expected that factoring out the first few of these nearby poles in each case will improve the apparent convergence in ϵ considerably.

In principle, having a method at hand that is capable of generating coefficients to very high accuracy, even to a couple of hundred digits, one could now use the algorithm PSLQ [16] combined with an educated guess of the number content of some of the yet-unknown constant terms, in order to search for analytic representations of the numerical results. These could then in turn be used as an inspiration to find useful transformations of the integral representation of the original integral, which might allow for a fully analytic solution in those cases where it could not yet be achieved. We have not made any attempts in that direction, since the numerical accuracy of the results Eqs. (8)-(25) should be sufficient for all practical purposes.

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References

- [1] K. Kajantie, M. Laine and Y. Schröder, Phys. Rev. D **65** (2002) 045008 [hep-ph/0109100].
- [2] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159; F. V. Tkachov, Phys. Lett. B **100** (1981) 65.
- [3] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033].
- [4] Y. Schröder, Nucl. Phys. Proc. Suppl. **116** (2003) 402 [hep-ph/0211288].
- [5] K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, Phys. Rev. D **67** (2003) 105008 [hep-ph/0211321].
- [6] M. Misiak and M. Münz, Phys. Lett. B **344** (1995) 308 [hep-ph/9409454].
- [7] Y. Schröder, hep-lat/0309112.
- [8] S. Laporta, Phys. Lett. B **549** (2002) 115 [hep-ph/0210336].
- [9] S. Laporta, Phys. Lett. B **523** (2001) 95 [hep-ph/0111123].
- [10] J. A. M. Vermaseren, math-ph/0010025.
- [11] A. K. Rajantie, Nucl. Phys. B **480** (1996) 729 [Erratum-ibid. B **513** (1998) 761] [hep-ph/9606216].
- [12] D. J. Broadhurst, hep-th/9806174.
- [13] E. Braaten and A. Nieto, Phys. Rev. D **51** (1995) 6990 [hep-ph/9501375].
- [14] A. Vuorinen, Master's Thesis, University of Helsinki (2001),
<http://ethesis.helsinki.fi/julkaisut/mat/fysii/pg/vuorinen/fourloop.pdf>
- [15] K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, JHEP **0304** (2003) 036 [hep-ph/0304048].
- [16] H. R. P. Ferguson, D. H. Bailey and S. Arno, Math. Comput. **68** (1999) 351.